

A GLOBAL APPROXIMATION RESULT BY AL TAYLOR AND THE STRONG OPENNESS CONJECTURE IN \mathbb{C}^n

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ABSTRACT. We improve a global approximation result by Al Taylor in \mathbb{C}^n for holomorphic functions in weighted Hilbert spaces. The main tools are a variation of the theorem of Hörmander on weighted L^2 -estimates for the $\bar{\partial}$ -equation together with the solution of the strong openness conjecture. A counterexample to a global strong openness conjecture in \mathbb{C}^n is also given here.

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1. INTRODUCTION

Let $z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$, $\|z\|^2 = |z_1|^2 + |z_2|^2 + \dots + |z_n|^2$. Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex domain. If φ is a measurable function in Ω , we denote by $L^2(\Omega, \varphi)$ the space of measurable functions f in Ω which are square integrable with respect to the measure $e^{-\varphi}d\lambda$, i.e.,

$$\|f\|_\varphi^2 := \int_\Omega |f|^2 e^{-\varphi} d\lambda < +\infty,$$

where $d\lambda$ is the Lebesgue measure. This is a subspace of the space $L^2(\Omega, \text{loc})$ of functions in Ω which are locally square integrable with respect to the Lebesgue measure. By $L_{p,q}^2(\Omega, \varphi)$ we denote the space of differential forms of type (p, q) with coefficients in $L^2(\Omega, \varphi)$. By $H(\Omega, \varphi)$ we denote the set of holomorphic functions on Ω which belong to $L^2(\Omega, \varphi)$. For a weight φ ,

let $H(\varphi)$ denote the space of entire functions f with finite φ norm, i.e., $\|f\|_{H(\varphi)}^2 = \int_{\mathbb{C}^n} |f|^2 e^{-\varphi} d\lambda \leq +\infty$.

In 1971 Al Taylor [6] investigated weighted approximation results for entire functions in \mathbb{C}^n . He proved:

Theorem 1.1. *Let $\varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \dots$ be plurisubharmonic (psh) functions on \mathbb{C}^n , let $\varphi = \lim_{j \rightarrow \infty} \varphi_j$, and suppose that $\int_K e^{-\varphi_1} d\lambda < \infty$ for every compact set K . Then the closure of $\bigcup_{j=1}^{\infty} H(\varphi_j + \log(1 + \|z\|^2))$ in the Hilbert space $L^2(\varphi + \log(1 + \|z\|^2))$ contains $H(\varphi)$.*

The condition on the sequence of plurisubharmonic functions resemble the condition in the recent work on the strong openness conjecture. Berndtsson proved the openness conjecture of Demailly-Kollár as follows.

Theorem 1.2 (Openness theorem). *Let U be a bounded pseudoconvex domain and $\varphi \in psh^-(U)$ ($psh^-(U)$ means the set of negative psh function on U) with*

$$\int_U e^{-\varphi} d\lambda < +\infty.$$

Let V be a relatively compact domain in U . Then there exists $p > 1$ such that

$$\int_V e^{-p\varphi} d\lambda < +\infty.$$

Slightly later, Guan-Zhou ([8] and [9]) proved a strong openness conjecture theorem as follows.

Theorem 1.3. *Let $\varphi_1 \leq \varphi_2 \leq \dots$, be negative plurisubharmonic functions on the unit polydisc $\Delta^n \subset \mathbb{C}^n$ so that $\varphi_n \nearrow \varphi$. Suppose that*

$$\int_{\Delta^n} |F|^2 e^{-\varphi} d\lambda < +\infty,$$

F is a holomorphic function on Δ^n . Then there exists a number $j \geq 1$, such that

$$\int_{\Delta_r^n} |F|^2 e^{-\varphi_j} d\lambda < +\infty,$$

for some $r \in (0, 1)$.

We remark that in this result, it suffices to assume that the φ_j are locally uniformly bounded above, rather than being negative. Also, the result is equivalently true if we add any fixed bounded measurable function to all the φ_j and the φ . Based on these results, we are able to improve Taylor's theorem as follows. We will prove

Theorem 1.4. *Let $\varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \dots$ be plurisubharmonic functions on \mathbb{C}^n . For any $\epsilon > 0$ let $\tilde{\varphi}_j = \varphi_j + \epsilon \log(1 + \|z\|^2)$ and $\tilde{\varphi} = \lim_{j \rightarrow +\infty} \tilde{\varphi}_j$. Then*

$\bigcup_{j=1}^{\infty} H(\tilde{\varphi}_j)$ is dense in $H(\tilde{\varphi})$.

The most direct generalization of the strong openness conjecture to \mathbb{C}^n is the following:

Conjecture 1.5. *Let $\varphi_1 \leq \varphi_2 \leq \dots$ be plurisubharmonic functions on \mathbb{C}^n which are locally uniformly bounded above. Let $\varphi = \lim_{k \rightarrow +\infty} \varphi_k$. If f is an entire function so that $\int_{\mathbb{C}^n} |f|^2 e^{-\varphi} d\lambda < \infty$, then for all large enough k , $\int_{\mathbb{C}^n} |f|^2 e^{-\varphi_k} d\lambda < \infty$.*

Nevertheless we have a counterexample for Conjecture 1.5 in \mathbb{C}^n . We will discuss it on the fourth part.

Recently, an approach based on Hörmander's L^2 -estimates of $\bar{\partial}$ to the openness conjecture was proposed by Chen [4], and we will borrow some techniques of his.

For the sake of convenience we always use C to represent universal constants in this paper.

2. WEIGHTED L^2 - ESTIMATE FOR THE $\bar{\partial}$ -EQUATION

Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain and let φ be a psh function on Ω . By Hörmander's L^2 -existence theorem for the $\bar{\partial}$ -equation (see [11]), we know that for every $\bar{\partial}$ -closed $(0, 1)$ -form v on Ω with $\int_{\Omega} |v|^2 e^{-\varphi} d\lambda < +\infty$, there exists a solution u to $\bar{\partial}u = v$ such that

$$\int_{\Omega} |u|^2 e^{-\varphi} d\lambda \leq C_{n, \text{diam}(\Omega)} \int_{\Omega} |v|^2 e^{-\varphi} d\lambda.$$

We say that u is the (unique) $L^2(\Omega, \varphi)$ -minimal solution of the $\bar{\partial}$ -equation if $u \perp \text{Ker} \bar{\partial}$ in $L^2(\Omega, \varphi)$, i.e., u has minimal norm $\|\cdot\|$ among all solutions.

In this paper we will first give the following main estimate by Hörmander [11] (see also [3]):

Theorem 2.1. *Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex domain. Let φ be a psh function on Ω satisfying*

$$i\partial\bar{\partial}\varphi \geq \Theta$$

in the sense of distributions for some continuous positive $(1, 1)$ -form Θ on Ω . For any $\bar{\partial}$ -closed $(0, 1)$ -form v with

$$\int_{\Omega} |v|_{\Theta}^2 e^{-\varphi} d\lambda < +\infty,$$

there exists $u \in L^2(\Omega, \text{loc})$ such that $\bar{\partial}u = v$ and

$$\int_{\Omega} |u|^2 e^{-\varphi} d\lambda \leq \int_{\Omega} |v|_{\Theta}^2 e^{-\varphi} d\lambda.$$

Here we use the notation: Let $\Theta = i \sum_{j,k} \Theta_{j,k} dz_j \wedge d\bar{z}_k$ be a continuous positive $(1,1)$ -form on Ω , i.e., the matrix $(\Theta_{j,k})$ is positive definite at every point. The pointwise norm of a $(0,1)$ -form v with respect to Θ is defined by

$$|v|_{\Theta}^2 := \sum_{j,k} \Theta^{j,k} v_j \bar{v}_k$$

where $(\Theta^{j,k})$ denotes the matrix inverse to $(\Theta_{j,k})$.

Remark. Actually it was Demailly who first gave the above formulation of Hörmander's estimate. The first remarkable variation of Hörmander's estimate is the following, [7],

Theorem 2.2 (Donnelly-Fefferman). *Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex domain and $\varphi \in \text{psh}(\Omega)$. Suppose ψ is a C^2 strictly psh function which satisfies*

$$(2.1) \quad r i \partial \bar{\partial} \psi \geq i \partial \psi \wedge \bar{\partial} \psi$$

for some $r > 0$. For each $\bar{\partial}$ -closed $(0,1)$ -form v there exists a solution u of $\bar{\partial}u = v$ satisfying

$$(2.2) \quad \int_{\Omega} |u|^2 e^{-\varphi} d\lambda \leq \text{const}_r \int_{\Omega} |v|_{i \partial \bar{\partial} \psi}^2 e^{-\varphi} d\lambda,$$

provided that the RHS of (2.2) is finite.

Later, Berndtsson [1] generalized theorem 2.2 as follows.

Theorem 2.3 (Berndtsson). *Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex domain and $\varphi \in \text{psh}(\Omega)$. ψ is a C^2 strictly psh function satisfying*

$$r i \partial \bar{\partial}(\varphi + \psi) \geq i \partial \psi \wedge \bar{\partial} \psi$$

in the sense of distributions for some $0 < r < 1$. Then for each $\bar{\partial}$ -closed $(0,1)$ -form v , there is a solution of $\bar{\partial}u = v$ which satisfies

$$(2.3) \quad \int_{\Omega} |u|^2 e^{\psi - \varphi} d\lambda \leq \frac{6}{(1-r)^2} \int_{\Omega} |v|_{\Theta}^2 e^{\psi - \varphi} d\lambda$$

for every continuous positive $(1,1)$ -form Θ with $i \partial \bar{\partial}(\varphi + \psi) \geq \Theta$ in the sense of distributions.

For the reader's convenience we include the proof of Theorem 2.3 here.

Proof. We note at first that it suffices to prove the Theorem in the case the right hand side of (2.3) is finite. So we suppose that v is a $\bar{\partial}$ -closed $(0,1)$ -form on Ω with $\int_{\Omega} |v|_{\Theta}^2 e^{\psi - \varphi} d\lambda < \infty$.

We exhaust Ω by a sequence of pseudoconvex domains $\Omega_j \subset\subset \Omega$ with smooth boundaries such that $\bar{\Omega}_j \subset \Omega_{j+1}$, $\Omega = \cup \Omega_j$. For each j , we may

choose smooth strictly psh function φ_j on Ω_{j+1} such that φ_j decrease monotonically to φ on Ω as $j \rightarrow \infty$ and

$$i\partial\bar{\partial}(\varphi_j + \psi) \geq \Theta$$

on $\bar{\Omega}_j$ for each continuous positive $(1,1)$ -form Θ with $i\partial\bar{\partial}(\varphi + \psi) \geq \Theta$. To see this, simply note that

$$i\partial\bar{\partial}(\varphi * \theta_\epsilon) = (i\partial\bar{\partial}\varphi) * \theta_\epsilon \rightarrow i\partial\bar{\partial}\varphi$$

on $\bar{\Omega}_j$ as $\epsilon \rightarrow 0$, where θ is a standard Friedrichs mollifier. It suffices to take $\varphi_j = \varphi * \theta_{\epsilon_j} + (|z|^2 + 1)/j$ with $\epsilon_j = \min\{d(\Omega_j, \partial\Omega), 1/j\} \ll 1$.

Since $i\partial\bar{\partial}(\varphi_j + \psi) \geq \Theta$, $i\partial\bar{\partial}(\varphi_j + \psi)$ is a continuous positive $(1,1)$ -form on Ω_j and ψ is bounded on Ω_j , we obtain that

$$(2.4) \quad \int_{\Omega_j} |v|_{i\partial\bar{\partial}(\varphi_j + \psi)}^2 e^{-(\psi + \varphi_j)} d\lambda \leq c_\psi \int_{\Omega_j} |v|_\Theta^2 e^{\psi - \varphi} d\lambda < \infty$$

since $|v|_{i\partial\bar{\partial}(\varphi_j + \psi)}^2 \leq |v|_\Theta^2$. Therefore Theorem 2.1 applies with $\varphi_j + \psi$ instead of φ . Hence there is a solution to $\bar{\partial}u_j = v$ on Ω_j so that

$$(2.5) \quad \int_{\Omega_j} |u_j|^2 e^{-\psi - \varphi_j} d\lambda < \infty.$$

Since ψ is bounded, $\int_{\Omega_j} |u_j|^2 e^{-\varphi_j} d\lambda < \infty$. We will assume that u_j is chosen as the unique solution which is perpendicular to the holomorphic functions, i.e. $\int_{\Omega_j} u_j \cdot \bar{g} e^{-\varphi_j} d\lambda = 0$ for all holomorphic functions on Ω with $\int_{\Omega_j} |g|^2 e^{-\varphi_j} d\lambda < \infty$.

Because of the boundedness of ψ we have $L^2(\Omega_j, \varphi_j) = L^2(\Omega_j, \varphi_j + \psi)$, and $u_j e^\psi \perp \text{Ker } \bar{\partial}$ in $L^2(\Omega_j, \varphi_j + \psi)$, i.e.,

$$\int_{\Omega_j} u_j e^\psi \cdot \bar{g} e^{-\psi - \varphi_j} d\lambda = 0$$

for any such g in $L^2(\Omega_j, \varphi_j + \psi)$. Thus $u_j e^\psi$ is the $L^2(\Omega_j, \varphi_j + \psi)$ -minimal solution of the equation

$$\bar{\partial} \tilde{u} = \bar{\partial}(u_j e^\psi).$$

It follows from Theorem 2.1 that

$$\begin{aligned}
\int_{\Omega_j} |u_j|^2 e^{\psi-\varphi_j} d\lambda &= \int_{\Omega_j} |u_j e^\psi|^2 e^{-\psi-\varphi_j} d\lambda \\
&\leq \int_{\Omega_j} |\bar{\partial}(u_j e^\psi)|_{i\bar{\partial}\bar{\partial}(\varphi_j+\psi)}^2 e^{-\psi-\varphi_j} d\lambda \\
&= \int_{\Omega_j} |v + \bar{\partial}\psi \wedge u_j|_{i\bar{\partial}\bar{\partial}(\varphi_j+\psi)}^2 e^{\psi-\varphi_j} d\lambda \\
&\leq (1 + \frac{1}{t}) \int_{\Omega_j} |v|_{i\bar{\partial}\bar{\partial}(\varphi_j+\psi)}^2 e^{\psi-\varphi_j} d\lambda \\
&\quad + (1+t) \int_{\Omega_j} |u_j|^2 |\bar{\partial}\psi|_{i\bar{\partial}\bar{\partial}(\varphi_j+\psi)}^2 e^{\psi-\varphi_j} d\lambda \\
&\leq (1 + \frac{1}{t}) \int_{\Omega_j} |v|_\Theta^2 e^{\psi-\varphi_j} d\lambda \\
&\quad + (1+t) \int_{\Omega_j} |u_j|^2 |\bar{\partial}\psi|_{\frac{1}{r}i\bar{\partial}\psi \wedge \bar{\partial}\psi}^2 e^{\psi-\varphi_j} d\lambda, \\
&\leq (1 + \frac{1}{t}) \int_{\Omega_j} |v|_\Theta^2 e^{\psi-\varphi_j} d\lambda \\
&\quad + (1+t)r \int_{\Omega_j} |u_j|^2 e^{\psi-\varphi_j} d\lambda,
\end{aligned}$$

in view of

$$i\bar{\partial}\bar{\partial}(\varphi_j + \psi) \geq \frac{1}{r}i\bar{\partial}\psi \wedge \bar{\partial}\psi$$

where $0 < t < 1$. Since

$$\int_{\Omega_j} |u_j|^2 e^{\psi-\varphi_j} d\lambda \leq \text{const}_\psi \int_{\Omega_j} |u_j|^2 e^{-\psi-\varphi_j} d\lambda < \infty$$

by (2.5), so we have

$$(2.6) \quad \int_{\Omega_j} |u_j|^2 e^{\psi-\varphi_j} d\lambda \leq \frac{1 + \frac{1}{t}}{1 - (1+t)r} \int_{\Omega_j} |v|_\Theta^2 e^{\psi-\varphi_j} d\lambda$$

provided $(1+t)r < 1$. It is not difficult to see that the coefficient in (2.6) attains the minimum $\frac{1}{(1-\sqrt{r})^2}$ when $t = \frac{1}{\sqrt{r}} - 1$.

Since $\{u_j\}$ is uniformly L^2 on each compact set of Ω , so we may choose a sequence by the standard diagonal sequence argument, which is still denoted by $\{u_j\}$ for the sake of simplicity, such that $u_j \rightarrow u$ weakly in $L^2(\Omega, \text{loc})$. For each fixed k , according to one corollary of weak convergence theorem

we have

$$\begin{aligned}
 \int_{\Omega_k} |u|^2 e^{\psi - \varphi_k} d\lambda &\leq \liminf_{j \rightarrow +\infty} \int_{\Omega_k} |u_j|^2 e^{\psi - \varphi_k} \\
 &\leq \liminf_{j \rightarrow +\infty} \int_{\Omega_j} |u_j|^2 e^{\psi - \varphi_j} \\
 &\leq \liminf_{j \rightarrow +\infty} \frac{6}{(1-r)^2} \int_{\Omega_j} |v|_{\Theta}^2 e^{\psi - \varphi_j} d\lambda \\
 &\leq \liminf_{j \rightarrow +\infty} \frac{6}{(1-r)^2} \int_{\Omega} |v|_{\Theta}^2 e^{\psi - \varphi} d\lambda \\
 &= \frac{6}{(1-r)^2} \int_{\Omega} |v|_{\Theta}^2 e^{\psi - \varphi} d\lambda,
 \end{aligned}$$

so that

$$\begin{aligned}
 \int_{\Omega} |u|^2 e^{\psi - \varphi} d\lambda &= \lim_{k \rightarrow +\infty} \int_{\Omega} \chi_{\Omega_k} \cdot |u|^2 e^{\psi - \varphi_k} d\lambda = \lim_{k \rightarrow +\infty} \int_{\Omega_k} |u|^2 e^{\psi - \varphi_k} d\lambda \\
 &\leq \frac{6}{(1-r)^2} \int_{\Omega} |v|_{\Theta}^2 e^{\psi - \varphi} d\lambda
 \end{aligned}$$

in view of the Lebesgue monotone convergence theorem. The proof is complete. \square

3. PROOF OF THEOREM 1.4

Our proof depends on L^2 -theory for the $\bar{\partial}$ -operator.

Proof of theorem 1.4. Notice that we can replace $\log(1 + \|z\|^2)$ by $\log(e + \|z\|^2)$ without changing the spaces and the norms because of equivalence.

Here we just give the proof when $0 < \varepsilon \leq 1$. The general case follows by first adding $(\varepsilon - 1) \log(e + \|z\|^2)$ to all φ_j and φ . Let $\chi : \mathbb{R} \rightarrow [0, 1]$ be a smooth function on \mathbb{C}^n satisfying $\chi|_{(-\infty, \frac{1}{2})} = 1$, $\chi|_{(1, +\infty)} = 0$ and $|\chi'| \leq 3$.

Set

$$\psi = -\log(\log(e + \|z\|^2)).$$

Then we have

$$i\partial\bar{\partial}\psi = -i \frac{\partial\bar{\partial}\log(e + \|z\|^2)}{\log(e + \|z\|^2)} + i \frac{\partial\log(e + \|z\|^2) \wedge \bar{\partial}\log(e + \|z\|^2)}{(\log(e + \|z\|^2))^2}$$

and

$$i\partial\psi \wedge \bar{\partial}\psi = i \frac{\partial\log(e + \|z\|^2) \wedge \bar{\partial}\log(e + \|z\|^2)}{(\log(e + \|z\|^2))^2}.$$

Thus

$$\begin{aligned}
& \frac{1}{2}i\partial\bar{\partial}(\tilde{\varphi}_j + \epsilon\psi) - i\epsilon\partial\frac{\psi}{2} \wedge \epsilon\bar{\partial}\frac{\psi}{2} \\
&= \frac{1}{2}i\partial\bar{\partial}\varphi_j + \frac{1}{2}i\epsilon\partial\bar{\partial}\log(e + \|z\|^2) + \frac{1}{2}i\epsilon\partial\bar{\partial}\psi - \frac{1}{4}i\epsilon^2\partial\psi \wedge \bar{\partial}\psi \\
&\geq \frac{1}{2}i\epsilon\partial\bar{\partial}\log(e + \|z\|^2) - \frac{1}{2}i\epsilon\frac{\partial\bar{\partial}\log(e + \|z\|^2)}{\log(e + \|z\|^2)} \\
&\quad + \frac{1}{2}i\epsilon\frac{\partial\log(e + \|z\|^2) \wedge \bar{\partial}\log(e + \|z\|^2)}{(\log(e + \|z\|^2))^2} \\
&\quad - \frac{1}{4}i\epsilon^2\frac{\partial\log(e + \|z\|^2) \wedge \bar{\partial}\log(e + \|z\|^2)}{(\log(e + \|z\|^2))^2} \\
&\geq \frac{1}{2}\epsilon\left(1 - \frac{1}{\log(e + \|z\|^2)}\right)i\partial\bar{\partial}\log(e + \|z\|^2) \\
&\quad + \frac{1}{2}\epsilon\left(1 - \frac{1}{2}\epsilon\right)i\frac{\partial\log(e + \|z\|^2) \wedge \bar{\partial}\log(e + \|z\|^2)}{(\log(e + \|z\|^2))^2} \\
&\geq 0
\end{aligned}$$

Let $f \in H(\tilde{\varphi})$, $N \in \mathbb{N}$ so that $\frac{1}{N} < \varepsilon$. Observe that

$$\left\{\frac{N}{2\varepsilon} \leq -\psi \leq \frac{N}{\varepsilon}\right\} = \{-\log 2 \leq \log(-\varepsilon\psi) + \log \frac{1}{N} \leq 0\} = A.$$

Then $f \cdot \chi(\log(-\varepsilon\psi) + \log \frac{1}{N})$ on A extends as a smooth function by setting it equal to f when $\log(-\varepsilon\psi) + \log \frac{1}{N} < -\log 2$ and 0 when $\log(-\varepsilon\psi) + \log \frac{1}{N} > 0$. The extension is a smooth function but it is not holomorphic on \mathbb{C}^n , so we modify it first. Put

$$v_N := f \cdot \bar{\partial}\chi\left(\log(-\varepsilon\psi) + \log \frac{1}{N}\right).$$

Apply Theorem 2.3 especially for $\Omega = \mathbb{C}^n$ with φ and ψ replaced by $\tilde{\varphi}_j + \frac{\epsilon\psi}{2}$ and $\frac{\epsilon\psi}{2}$ respectively, $r = \frac{1}{2}$ and $\Theta = i\epsilon\partial\psi \wedge \epsilon\bar{\partial}\psi$, we then obtain a solution $u_{j,N}$ of $\bar{\partial}u = v_N$ on \mathbb{C}^n satisfying

$$\begin{aligned}
& \int_{\mathbb{C}^n} |u_{j,N}|^2 e^{\frac{\epsilon\psi}{2} - (\tilde{\varphi}_j + \frac{\epsilon\psi}{2})} d\lambda \leq \frac{6}{(1 - \frac{1}{2})^2} \int_{\mathbb{C}^n} |f\bar{\partial}\chi|_{\Theta}^2 e^{\frac{\epsilon\psi}{2} - (\tilde{\varphi}_j + \frac{\epsilon\psi}{2})} d\lambda \\
&\leq 24 \cdot 9 \int_{\frac{N}{2\varepsilon} \leq -\psi \leq \frac{N}{\varepsilon}} |f|^2 \frac{1}{(\epsilon\psi)^2} |\epsilon\bar{\partial}\psi|_{\epsilon^2 i\partial\psi \wedge \bar{\partial}\psi}^2 e^{-\tilde{\varphi}_j} d\lambda \\
&\leq \frac{C}{N^2} \int_{\frac{N}{2\varepsilon} \leq -\psi \leq \frac{N}{\varepsilon}} |f|^2 e^{-\tilde{\varphi}_j} d\lambda.
\end{aligned}$$

Put $K := \{z : z \in \mathbb{C}^n, -\psi \leq \frac{N}{\varepsilon}\}$. Let $q \in K$ and let $\Delta(q)$ be a polydisc centered at q . Since $f \in H(\tilde{\varphi})$, we have that $\int_{\Delta(q)} |f|^2 e^{-\tilde{\varphi}} d\lambda < \infty$.

By Theorem 1.3 there exists a $j_q \geq 1$ and a number $0 < r_q < 1$ so that $\int_{\Delta(q)_{r_q}} |f|^2 e^{-\tilde{\varphi}_{j_q}} d\lambda < \infty$.

By compactness there are finitely many $q_i \in K, j_i \in \mathbb{N}, 1 \leq i \leq m$ so that $K \subset \cup_{i=1}^m \Delta_{r_i}(q_i)$ and $\int_{\Delta_{r_i}(q_i)} |f|^2 e^{-\tilde{\varphi}_{j_i}} d\lambda < \infty$. Let $j_0 = \max\{j_i\}$. Then $\int_K |f|^2 e^{-\tilde{\varphi}_{j_0}} d\lambda < \infty$.

But $-\tilde{\varphi}_j$ monotonically decreases to $-\tilde{\varphi}$, $|f|^2 e^{-\tilde{\varphi}_{j_0}}$ can be seen as the control function on K . So we have for all large enough $j \geq j_0$,

$$\int_K |f|^2 e^{-\tilde{\varphi}_j} d\lambda \longrightarrow \int_K |f|^2 e^{-\tilde{\varphi}} d\lambda$$

in view of the Lebesgue dominated convergence theorem. Set

$$F_{j,N} = f \cdot \chi \left(\log(-\epsilon\psi) + \log \frac{1}{N} \right) - u_{j,N}.$$

We then have $F_{j,N} \in \mathcal{O}(\mathbb{C}^n)$ such that for each $j \geq j_0 \gg 1$,

$$\begin{aligned} \|F_{j,N}\|_{H(\tilde{\varphi}_j)} &\leq \left\| f \cdot \chi \left(\log(-\psi) + \log \frac{1}{N} \right) \right\|_{H(\tilde{\varphi}_j)} + \|u_{j,N}\|_{H(\tilde{\varphi}_j)} \\ &\leq \left(\int_{-\psi < \frac{N}{\epsilon}} |f|^2 e^{-\tilde{\varphi}_j} d\lambda \right)^{\frac{1}{2}} + \left(\int_{\mathbb{C}^n} |u_{j,N}|^2 e^{-\tilde{\varphi}_j} d\lambda \right)^{\frac{1}{2}} \\ &\leq \left(1 + \frac{C}{N} \right) \|f\|_{H(\tilde{\varphi})} < +\infty. \end{aligned}$$

Thus $F_{j,N} \in \bigcup_{j=1}^{\infty} H(\tilde{\varphi}_j)$.

On the other hand, we have

$$\begin{aligned} &\int_{\mathbb{C}^n} |F_{j,N} - f|^2 e^{-\tilde{\varphi}} d\lambda \\ &= \int_{\mathbb{C}^n} \left| f \cdot \chi \left(\log(-\epsilon\psi) + \log \frac{1}{N} \right) - u_{j,N} - f \right|^2 e^{-\tilde{\varphi}} d\lambda \\ &\leq 2 \int_{\mathbb{C}^n} \left| f \cdot \chi \left(\log(-\epsilon\psi) + \log \frac{1}{N} \right) - f \right|^2 e^{-\tilde{\varphi}} d\lambda + 2 \int_{\mathbb{C}^n} |u_{j,N}|^2 e^{-\tilde{\varphi}_j} d\lambda \\ &\leq 2 \int_{-\psi \geq \frac{N}{2\epsilon}} |f|^2 e^{-\tilde{\varphi}} d\lambda + \frac{C}{N^2} \|f\|_{H(\tilde{\varphi})}^2 \longrightarrow 0 \quad (N \longrightarrow +\infty) \end{aligned}$$

Thus $\bigcup_{j=1}^{\infty} H(\tilde{\varphi}_j)$ is dense in $H(\tilde{\varphi})$, which completes the proof.

□

4. A COUNTEREXAMPLE FOR STRONG OPENNESS CONJECTURE TO \mathbb{C}^n

We show that Conjecture 1.5 is however false.

Use the notation $|z|^{2|\alpha|} = |z_1|^{2\alpha_1} \cdots |z_n|^{2\alpha_n}$.

Lemma 4.1. *Let z^α be a holomorphic monomial. The following are equivalent in \mathbb{C}^n*

1. $\int_{\|z\|>1} \frac{|z|^{2|\alpha|}}{\|z\|^N} d\lambda < \infty$;
2. $N \geq 2|\alpha| + 2n + 1$.

Proof. We prove that 1 implies 2.

For $k = 1, 2, \dots$, let $A_k = \{z; 2^k \leq |z_i| \leq 2^{k+1}, \forall i = 1, \dots, n\}$. On A_k we have:

- a) $|z|^{2|\alpha|} \geq (2^k)^{2|\alpha|}$;
- b) $\|z\|^N \leq (\sqrt{n}(2^{k+1}))^N$;
- c) The volume $|A_k| \geq (\pi(2^k)^2)^n$.

Hence we get

$$\begin{aligned}
 \infty &> \int_{\|z\|>1} \frac{|z|^{2|\alpha|}}{\|z\|^N} d\lambda \\
 &> \sum_k \int_{A_k} \frac{|z|^{2|\alpha|}}{\|z\|^N} d\lambda \\
 &> \sum_k \frac{(2^k)^{2|\alpha|} (\pi(2^k)^2)^n}{(\sqrt{n}(2^{k+1}))^N} \\
 &= c_N \sum_k 2^{k(2|\alpha|+2n-N)}
 \end{aligned}$$

This implies that $2|\alpha| + 2n - N \leq -1$. Hence 2 follows.

Next we assume 2. Observe that $|z|^{2|\alpha|} \leq \|z\|^{2|\alpha|}$. We then get

$$\int_{\|z\|>1} \frac{|z|^{2|\alpha|}}{\|z\|^N} d\lambda \leq \int_{\|z\|\geq 1} \frac{1}{\|z\|^{2n+1}} d\lambda < \infty. \quad \square$$

Lemma 4.2. *Let ψ be a plurisubharmonic function on \mathbb{C}^n . Suppose that there exist positive number R, C, N so that $\psi(z) = C + N \log \|z\|$ for all $z, \|z\| \geq R$. Then if $\int_{\mathbb{C}^n} |f|^2 e^{-\psi} d\lambda < \infty$, then f is a polynomial of degree at most $\frac{N-2n-1}{2}$.*

Proof. We write f in power series, $f = \sum_{\alpha} a_{\alpha} z^{\alpha}$. We have that

$$\int_{\|z\|\geq R} \sum_{\alpha} |a_{\alpha}|^2 |z|^{2\alpha} \frac{e^{-C}}{\|z\|^N} d\lambda \leq \infty.$$

In particular each individual term in the integral must be finite. By the above Lemma the only non zero terms can be those for which $N \geq 2|\alpha| + 2n + 1$, i.e. $|\alpha| \leq \frac{N-2n-1}{2}$. \square

For $k = 1, 2, \dots$, let N_k be an increasing sequence so that $N_k \geq 2n+1+2k$. We define inductively continuous plurisubharmonic functions, $\varphi_1 \leq \varphi_2 \leq \dots$ and $\varphi_j \rightarrow \varphi$.

$$\begin{aligned}
\varphi_1 &= \max\{1, \ln \|z\|\} \\
\varphi_2 &= \varphi_1, \|z\| \leq 2 \\
\varphi_2 &= C_2 + N_2 \log \|z\|, \|z\| \geq 2 \\
&\dots \\
\varphi_{k+1} &= \varphi_k, \|z\| \leq k+1 \\
\varphi_{k+1} &= C_{k+1} + N_{k+1} \log \|z\|, \|z\| \geq k+1
\end{aligned}$$

We see that

Lemma 4.3. *The limit $\varphi = \lim_k (C_k + N_k \log \|z\|)$, $k \leq \|z\| \leq k+1$, $k = 2, \dots$, $\varphi = \max\{1, \log \|z\|\}$ when $\|z\| \leq 2$.*

Let $A_\ell := \int_{\|z\| \leq \ell} e^{-\varphi_1} |z_1|^{2\ell} d\lambda$ and set $B_\ell := \int_{\|z\| \geq \ell} |z_1|^{2\ell} e^{-\varphi_\ell} d\lambda$. Pick ϵ_ℓ so small that $\epsilon_\ell(A_\ell + B_\ell) < 1/2^\ell$.

Lemma 4.4. *The function $f = \sum_{k \geq 1} \epsilon_k z_1^k$ belongs to $H(\varphi)$ but not to any $H(\varphi_k)$.*

Proof. We only need to show that $\int_{\mathbb{C}^n} |f|^2 e^{-\varphi} d\lambda < \infty$. So we need to show that $\sum_k \int_{\mathbb{C}^n} \epsilon_k |z_1|^{2k} e^{-\varphi} d\lambda < \infty$. We get

$$\begin{aligned}
\int_{\mathbb{C}^n} \epsilon_k |z_1|^{2k} e^{-\varphi} d\lambda &= \int_{\|z\| \leq k} \epsilon_k |z_1|^{2k} e^{-\varphi} d\lambda + \int_{\|z\| \geq k} \epsilon_k |z_1|^{2k} e^{-\varphi} d\lambda \\
&\leq \epsilon_k (A_k + B_k) \\
&\leq 1/2^k.
\end{aligned}$$

□

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